GIRRAWEEN HIGH SCHOOL
MATHEMATICS
YEAR 12 Extension 1 HSC
Task 1, 2014

Time Allowed: 90 minutes **Name:**______________________

Instructions: Examiner: C. McMillan

- Attempt all questions
- Fill in the circle on the multiple choice answer sheet next to the best response for the questions in Part A
- Start each question in Part B on a new page
- All necessary working must be shown
- Marks may be deducted for careless or badly arranged work

PART A (5 marks)

For questions 1-5 circle the best response from the following:

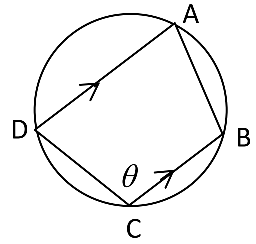
Question 1: The coefficient of x^4 in the expansion $\left(x-\sqrt{2}\right)^6$ is:

A) -30 B) 30 C) -60 D) 60

Question 2: The middle term of the expansion $\left(1+\frac{3x}{4}\right)^6$ is:

A) $540x^3$ B) $\frac{1215}{4^4}x^4$ C) $\frac{540}{64}x^3$ D) $1215x^4$

Question 3: If $\angle BCD$ is equal to θ then $\angle ABC$ is equal to:



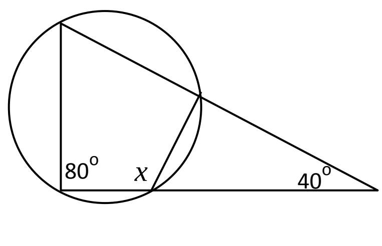
A) $180-\theta$ B) θ C) $180+\theta$ D) 180

Question 4: If $n=k$ is $1+2+4+...+2^{k-1}=2^k-1$ then $n=k+1$ is:

A) $1+2+4+...+2^{k-1}+2^{k+2}=2^{k-1}-1$ B) $1+2+4+...+2^{k-1}+2^{k-2}=2^{k+1}-1$

C) $1+2+4+...+2^{k-1}+2^k=2^{k+1}-1$ D) $1+2+4+...+2^{k-1}+2^k=2^k-1$

Question 5: The value of x is:



A) 120° B) 80° C) 60° D) 40°

PART B

Question 6 (13 marks)

(a) Prove by mathematical induction that for all positive integers

$1+4+7+\ldots+(3n-2)=\dfrac{n(3n-1)}{2}$. **(5)**

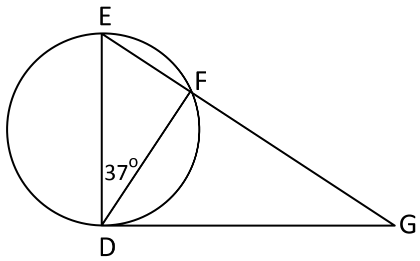
(b) Using mathematical induction prove that for all positive integers

$7^{2n}-3^{3n}$ is divisible by 11. **(4)**

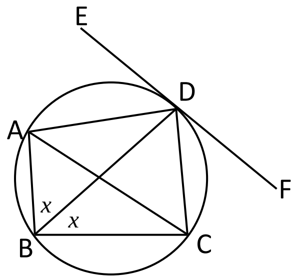
(c) Show by the principle of mathematical induction that $n!>2^n$ for $n>3$. **(4)**

Question 7 (16 marks) **Draw all diagrams in your answer booklet.**

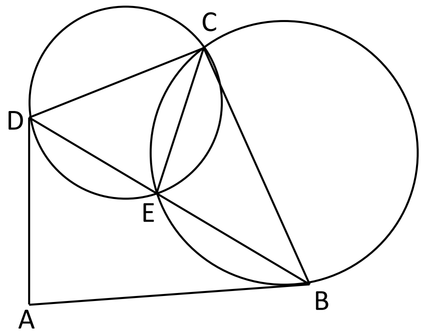
(a) DE is a diameter of the circle. DG is a tangent. If $\angle EDF = 37^o$ calculate the size of $\angle EGD$, giving reasons. **(4)**



(b) The diagonal BD of the quadrilateral ABCD, bisects $\angle ABC$. Prove that diagonal AC is parallel to the tangent at D. **(6)**



(c) AB and AD are tangents to the circles. DEB is a straight line. Prove that ABCD is a cyclic quadrilateral. **(6)**



Question 8 (13 marks)

(a) Find the 4th term in the expansion $\left(\frac{m}{2}+3n\right)^8$. **(2)**

(b) Find the coefficient of x^5 in the expansion of $(2-x)^7$. **(2)**

(c) In the expansion $\left(x-\frac{1}{x}\right)^6\left(x+\frac{1}{x}\right)^8$, find the coefficient of x^{-8}. **(4)**

(d) In the expansion $(2+3x)^n$, the coefficients of x^3 and x^4 are in the ratio 8:15.

Find n. **(5)**

Question 9 (15 marks)

(a) Find in the expansion $(5+2x)^{12}$:

i) The ratio $\frac{T_{k+1}}{T_k}$, showing all necessary working. **(4)**

ii) The greatest coefficient. **(2)**

iii) The greatest term when $x=\frac{1}{2}$. **(3)**

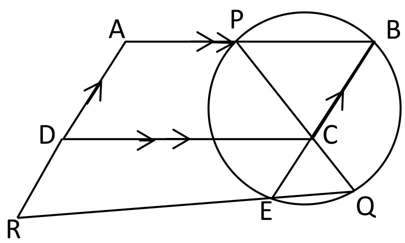
(b) Find n if the coefficients of the second, third and fourth terms in the expansion of $(1+x)^n$ are successive terms of an arithmetic series. **(6)**

Question 10 (14 marks) **Draw diagrams in your answer booklet**

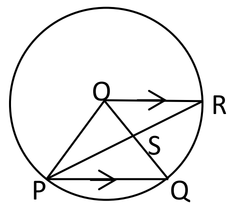
(a) ABCD is a parallelogram. AD and QE are produced to meet at R.
PQ and BE are chords.

(i) Prove that APQR is a cyclic quadrilateral. **(6)**

(ii) Hence, show that $\angle PCD = \angle DRQ$. **(2)**



(b) PQ is a chord of the circle with centre O. OR is parallel to PQ. Prove that $\angle RSQ$ is three times the size of $\angle RPQ$. **(6)**



END OF EXAMINATION ☺

YEAR 11 HSC Ext 1 Task 1 2014.

Q1 B

Q2 C

Q3 B.

Q4 C

Q5 A.

Question 6.

a) $1+4+7+\ldots+(3n-2)=\frac{n(3n-1)}{2}$.

Show true for $n=1$.

$LHS = 1 \qquad RHS = \frac{1(3-1)}{2}$

$= 1.$

$\therefore$ True for $n=1$.

Assume true for $n=k$.

$\therefore 1+4+7+\ldots+(3k-2)=\frac{k(3k-1)}{2}$.

Prove true for $n=k+1$

$\therefore 1+4+7+\ldots+(3k-2)+(3k+1)$

$=\frac{(k+1)(3k+2)}{2}$.

$LHS = 1+4+7+\ldots+(3k-2)+(3k+1)$

$= \frac{k(3k-1)}{2} + (3k+1)$

$= \frac{1}{2}[k(3k-1)+6k+2]$.

$= \frac{1}{2}(3k^2-k+6k+2)$

$= \frac{1}{2}(3k^2+5k+2)$

$= \frac{1}{2}(k+1)(3k+2)$

$= RHS$.

$\therefore$ If true for $n=k$ then true for $n=k+1$.

$\therefore$ By the principle of mathematical induction true for all $n \geq 1$.

b) $7^{2n}-3^{3n}$ $(\div 11.)$.

Show true for $n=1$

7^2-3^3

$=49-27$

$=22$.

$\therefore$ True for $n=1$.

Assume true for $n=k$.

$7^{2k}-3^{3k}=11P$ (P is an integer)

$7^{2k}=11P+3^{3k}$

Show true for $n=k+1$.

$7^{2(k+1)}-3^{3(k+1)}=11Q$

(Q is an integer)

$LHS = 7^{2k+2}-3^{3k+3}$.

$= 7^2\times 7^{2k} - 3^3\times 3^{3k}$.

$= 49(11P+3^{3k}) - 3^3\times 3^{3k}$.

$= 49\times 11P + 49\times 3^{3k} - 27\times 3^{3k}$

$= 49\times 11P + 22\times 3^{3k}$

$= 11(49P + 2\times 3^{3k})$

$= 11Q$.

$\therefore$ If true for $n=k$ then true for $n=k+1$.

$\therefore$ By the principle of mathematical induction true for all $n \geq 1$.

Question 6 cont.

c) $n! > 2^n \quad n > 3$.

Show true for $n = 4$.

$LHS = 4! \quad RHS = 2^4$

$= 24 \quad = 16$.

$\therefore 24 > 16$

$\therefore$ True for $n = 4$.

Assume true for $n = k$.

$\therefore k! > 2^k$

Prove true for $n = k+1$.

$\therefore (k+1)! > 2^{k+1}$.

$LHS = (k+1)!$

$= (k+1)k!$

$> (k+1) \times 2^k$

$> 2 \times 2^k$ (as $k > 3$)

$= 2^{k+1}$.

$\therefore (k+1)! > (k+1)2^k > 2^{k+1}$.

$\therefore$ If true for $n = k$ then true for $n = k+1$

$\therefore$ By the principle of mathematical induction true for all $n > 3$.

Question 7

a) $\angle FDG = 90 - 37$

(tangent $\perp$ to radius at point of contact)

$= 53°$.

$\angle DEF = 53°$

($\angle$ in the alternate segment)

$\angle EGD = 180 - (53 + 90)$

($\angle$ sum of a Δ)

$= 37°$.

b) $\angle EDA = x$

($\angle$ in the alternate segment)

$\angle FDC = x$.

($\angle$ in the alternate segment)

$\angle DCA = x$

($\angle$ in the alternate segment)

$\angle DAC = x$

($\angle$ in the alternate segment)

$\therefore \angle EDA = \angle DAC = x$.

(alternate $\angle$'s equal)

$\therefore EF \parallel AC$.

c) Let $\angle ADB = x$, $\angle ABE = y$.

$\angle DCE = x$.

($\angle$ in the alternate segment)

$\angle ECB = y$.

($\angle$ in the alternate segment)

$\angle DAB = 180 - (x+y)$

($\angle$ sum of a Δ)

$\therefore \angle DAB + \angle DCB = 180 - (x+y) + (x+y)$

$= 180$

(supplementary)

$\therefore$ ABCD is a cyclic quadrilateral as opposite angles are supplementary.

Question 8.

a) $T_4 = {}^8C_3 \left(\frac{m}{2}\right)^5 (3n)^3$

$= {}^8C_3 \times \frac{m^5}{32} \times 27n^3.$

$= \frac{27}{32} \times {}^8C_3\ m^5 n^3.$

b) ${}^7C_5\ 2^2 (-x)^5.$

$= -84x^5$

$\therefore$ The coefficient is -84.

c) $\left(x - \frac{1}{x}\right)^6.$

$= {}^6C_0 x^6 - {}^6C_1 x^4 + {}^6C_2 x^2 - {}^6C_3 + {}^6C_4 x^{-2}$

$- {}^6C_5 x^{-4} + {}^6C_6 x^{-6}.$

$\left(x + \frac{1}{x}\right)^8$

$= {}^8C_0 x^8 + {}^8C_1 x^6 + {}^8C_2 x^4 + {}^8C_3 x^2$

$+ {}^8C_4 + {}^8C_5 x^{-2} + {}^8C_6 x^{-4} + {}^8C_7 x^{-6} + {}^8C_8 x^{-8}$

$\therefore$ Coefficient x^{-8}

$= \left({}^8C_5 \times {}^6C_6\right) + \left({}^8C_6 \times \left(-{}^6C_5\right)\right)$

$+ \left({}^8C_7 \times {}^6C_4\right) + \left({}^8C_8 \times \left(-{}^6C_3\right)\right)$

$= 56 - 168 + 120 - 20.$

$= -12.$

$\therefore$ The coefficient is -12.

Question 8. cont.

d) $T_4 = {}^nC_3\ 2^{n-3} (3x)^3.$

$T_5 = {}^nC_4\ 2^{n-4} (3x)^4$

$\frac{{}^nC_3\ 2^{n-3}\ 3^3}{{}^nC_4\ 2^{n-4}\ 3^4.} = \frac{8}{15}.$

$\frac{{}^nC_3}{{}^nC_4} \times \frac{2}{3} = \frac{8}{15}.$

$\frac{4}{n-3} \times \frac{2}{3} = \frac{8}{15}.$

$\therefore \frac{8}{3(n-3)} = \frac{8}{15}.$

$\therefore 3(n-3) = 15.$

$n - 3 = 5.$

$\therefore n = 8.$

Question 9

a) $(5+2x)^{12}.$

i) $T_{k+1} = {}^{12}C_k\ 5^{12-k} (2x)^k.$

$T_k = {}^{12}C_{k-1}\ 5^{13-k} (2x)^{k-1}.$

$\therefore \frac{T_{k+1}}{T_k} = \frac{{}^{12}C_k\ 5^{12-k}\ 2^k x^k}{{}^{12}C_{k-1}\ 5^{13-k}\ 2^{k-1} x^{k-1}}$

$= \frac{\frac{12!}{k!(12-k)!}}{\frac{12!}{(k-1)!(13-k)!}} \times \frac{2}{5} x.$

Question 9 cont.

ii) $= \dfrac{(13+k)}{k} \times \dfrac{2}{5} x$

$= \dfrac{2(13-k)}{5k} x$

$\dfrac{26-2k}{5k} > 1$

$5k < 26 - 2k$

$k < 3.7 \quad \therefore k = 3.$

$\therefore$ The greatest coefficient is.

$T_4 = {}^{12}C_3 \, 5^9 \, 2^3$

iii) $\dfrac{2(13-k)x}{5k} > 1.$

When $x = \frac{1}{2}$.

$\dfrac{2(13-k)\frac{1}{2}}{5k} > 1.$

$13 - k > 5k$

$13 > 6k$

$\therefore k < 2.1$

$\therefore k = 2.$

$T_3 = {}^{12}C_2 \, 5^{10} \, 2^2 \left(\frac{1}{2}\right)^2$

$= {}^{12}C_2 \, 5^{10}.$

c) $T_2 = {}^nC_1 \, x \quad T_3 = {}^nC_2 \, x^2$

$T_4 = {}^nC_3 \, x^3.$

$T_3 - T_2 = T_4 - T_3.$

${}^nC_2 - {}^nC_1 = {}^nC_3 - {}^nC_2.$

$\dfrac{n!}{2!(n-2)!} - \dfrac{n!}{(n-1)!} = \dfrac{n!}{3!(n-3)!} - \dfrac{n!}{2!(n-2)!}$

$\dfrac{n!(n-3)}{2!(n-1)!} = \dfrac{n!(n-5)}{3!(n-2)!}$

$\therefore 3n!(n-3) = n!(n-1)(n-5).$

$3(n-3) = (n-1)(n-5)$

$3n - 9 = n^2 - 6n + 5$

$n^2 - 9n + 14 = 0$

$(n-7)(n-2) = 0.$

$\therefore n = 7.$

Question 10.

a) i) Let $\angle PCD = x$.

$\angle BPC = x$

(alternate $\angle$'s, $AB \parallel DC$)

Let $\angle PBE = y$. (6)

$\angle PQE = y$.

($\angle$'s subtended at circumference by arc PE)

$\angle BAD = 180 - y$

(co-interior $\angle$'s, $BC \parallel AD$).

$\therefore \angle BAD + \angle PQE = 180$.

(supplementary).

$\therefore$ APQR is a cyclic quadrilateral.

(Opposite $\angle$'s are supplementary)

ii) $\angle APC = 180 - x$

(straight $\angle$) 1

$\therefore \angle DRQ = x$

(opposite $\angle$'s of a cyclic quadrilateral) 1

$\therefore \angle PCD = \angle DRQ$.

b) Let $\angle RSQ = y$

and $\angle RPQ = x$.

(6)

$\angle ROQ = 2x$.

($\angle$ at the centre is twice $\angle$ at the circumference on the same arc)

$\angle ORS = x$

(alternate $\angle$'s, $OR \parallel PQ$.

$\therefore \angle RSQ = 2x + x$

(exterior $\angle$ of a Δ)

$= 3x$

$\therefore y = 3x$.